Algebra I | 80 Points

Notes.
(a) This exam is for the duration of two and a half hours only.
(b) Justify all your steps. You may use any result proved in class unless you have been asked to prove the same.
(c) $\mathbb{N}$ = natural numbers, $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers.
(d) $\mathbb{Z}_n$ = the set of integers modulo n, $(\mathbb{Z}_n)^\times$ = the set of those integers in $\mathbb{Z}_n$ that are coprime to n.
(e) For any element a in a group G, $o(a)$ denotes the order of the element.

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1. [12 points] Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a function such that for every $x \in \mathbb{R}$, $f^{-1}(x)$ is a countable set (possibly empty). Prove that there are uncountably many $\alpha \in \mathbb{R}$ such that $f(\alpha)$ is irrational.

2. [12 points] Prove or disprove the following statement: *Any total order on $\mathbb{N}$ is a well order.* This means you should either provide a proof of the above statement or disprove it by giving an example of a total order on $\mathbb{N}$ which is not a well order.

3. [12 points] Verify that 293 is invertible in $(\mathbb{Z}_{929})^\times$ and find its multiplicative inverse.

4. [12 points] Let a_n denote the number of elements of the set $\{\sigma \in S_7 \mid o(\sigma) = n\}$ where S_7 is the symmetric group on 7 symbols. Calculate a_2, a_6, a_8 and a_{12}.

5. [16 points]
   (i) If every element of a group G has order 2, prove that G is abelian.
   (ii) Give an example of a group G and two elements $a, b \in G$ such that $o(a) = 2 = o(b)$ while $o(ab) = 6$.

6. [16 points] Let G be a cyclic group of order n. Prove that every subgroup of G is cyclic and prove that for every positive integer d dividing n, there is a unique subgroup of order d.